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Approximating Solutions of Maximal Monotone Operators in Hilbert Spaces

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Let H be a real Hilbert space and let $T: H \to 2^H$ be a maximal monotone operator. In this paper, we first introduce two algorithms of approximating solutions of maximal monotone operators. One of them is to generate a strongly convergent sequence with limit $v \in T^{-1}0$. The other is to discuss the weak convergence of the proximal point algorithm. Next, using these results, we consider the problem of finding a minimizer of a convex function. Our methods are motivated by Halpern's iteration and Mann's iteration. © 2000 Academic Press

Key Words: maximal monotone operator; resolvent; proximal point algorithm; iteration; strong convergence; weak convergence.

1. INTRODUCTION

Let H be a real Hilbert space and let $T: H \to 2^H$ be a maximal monotone operator. Then the problem of finding a solution $v \in H$ with $0 \in Tv$ has been investigated by many researchers; see, for example, Bruck [3], Rockafellar [14], Brézis and Lions [2], Reich [12, 13], Nevanlinna and Reich [11], Bruck and Reich [4], Takahashi and Ueda [16], Jung and Takahashi [7], Khang [8] and others. One popular method of solving $0 \in Tv$ is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_0 = x \in H$, a sequence $\{x_n\}$ in H by the rule

$$x_{n+1} = J_{r_n} x_n, \qquad n = 0, 1, 2, ..., \tag{1.1}$$

where $J_{r_n} = (I + r_n T)^{-1}$ and $\{r_n\}$ is a sequence of positive real numbers. Some of them dealt with the weak convergence of the sequence $\{x_n\}$ generated by (1.1) and others proved strong convergence theorems by imposing strong assumptions on T. On the other hand, Wittmann [18]

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and Mann [9] considered the following iterative schemes for finding a fixed point of a nonexpansive mapping U of H into itself,

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\, U x_n, \qquad n = 0, 1, 2, ... \tag{1.2}$$

and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\, U x_n, \qquad n = 0, 1, 2, ..., \tag{1.3}$$

respectively, where $x_0 = x \in H$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$; see originally Halpern [6] for (1.2). Wittmann proved that if the set $F(U)$ of fixed points of U is nonempty, then the sequence $\{x_n\}$ generated by (1.2) converges strongly to some $z \in F(U)$. Mann also proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to some $z \in F(U)$.

In this paper, motivated by (1.2) and (1.3), we introduce the following two iterative schemes,

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\, J_{r_n} x_n, \qquad n = 0, 1, 2, ... \tag{1.4}$$

and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\, J_{r_n} x_n, \qquad n = 0, 1, 2, ..., \tag{1.5}$$

where $x_0 = x \in H$, $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{r_n\}$ is a sequence in $(0, \infty)$. Then we show that the sequence $\{x_n\}$ generated by (1.4) converges strongly to some $v \in T^{-1}0$ and the sequence $\{x_n\}$ generated by (1.5) converges weakly to some $v \in T^{-1}0$. Further, using these results, we investigate two algorithms in the case of $T = \partial f$, where f is a proper lower semicontinuous convex function.

2. PRELIMINARIES

Throughout this paper, we denote the set of all nonnegative integers by $\mathbb{N}$. All results in this paper are set in a real Hilbert space H with norm $\|\cdot\|$ and inner product $\langle\ ,\ \rangle$. When $\{x_n\}$ is a sequence in H, we denote strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)\, y\|^2 = \lambda\, \|x\|^2 + (1-\lambda)\, \|y\|^2 - \lambda(1-\lambda)\, \|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. A mapping $U\colon H \to H$ is said to be nonexpansive if $\|Ux - Uy\| \leqslant \|x - y\|$ for all $x, y \in H$. We denote the set of all fixed points of U by $F(U)$. A multivalued operator $T: H \to 2^H$ with domain $D(T) = \{z \in H : Tz \neq \emptyset\}$ and range $R(T) = \bigcup \{Tz : z \in D(T)\}$ is said to be monotone if for each $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \geqslant 0$. A monotone operator T is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. Let I denote the identity operator on H and let $T: H \to 2^H$ be a maximal monotone operator. Then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r: H \to H$ by $J_r = (I + rT)^{-1}$. It is called the resolvent (or the proximal mapping) of T. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in TJ_r x$ and $\|A_r x\| \leqslant \inf\{\|y\| : y \in Tx\}$ for all $x \in H$. We also know that $T^{-1}0 = F(J_r)$ for all $r > 0$; see, for instance, Rockafellar [14] or Takahashi [15]. It is shown in Rockafellar [14, Proposition 1] that

$$\|J_r x - J_r y\|^2 + r^2 \|A_r x - A_r y\|^2 \leqslant \|x - y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $r > 0$. Let $f: H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then we can define the subdifferential ∂f of f by

$$\partial f(x) = \{z \in H : f(y) \geqslant f(x) + \langle y - x, z \rangle \text{ for all } y \in H\}$$

for all $x \in H$. It is well known that ∂f is a maximal monotone operator of H into itself; see Minty [10].

3. STRONG CONVERGENCE THEOREM

Let $T: H \to 2^H$ be a maximal monotone operator and let $J_r: H \to H$ be the resolvent of T for each $r > 0$. Then we consider the following algorithm. The sequence $\{x_n\}$ is generated by

$$\begin{cases} x_0 = x \in H, \\ y_n \approx J_{r_n} x_n, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)\, y_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. Here the criterion for the approximate computation of y_n in (3.1) will be

$$\|y_n - J_{r_n} x_n\| \leqslant \delta_n, \tag{3.2}$$

where $\sum_{n=0}^{\infty} \delta_n < \infty$. Motivated by Wittmann [18], we obtain the following theorem.

THEOREM 1. *Let* $T: H \to 2^H$ *be a maximal monotone operator. Let* $x \in H$ *and let* $\{x_n\}$ *be a sequence generated by* (3.1) *under criterion* (3.2), *where* $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} r_n = \infty$. *If* $T^{-1}0 \neq \emptyset$, *then* $\{x_n\}$ *converges strongly to* Px, *where* P *is the metric projection of* H *onto* $T^{-1}0$.

Proof. From $T^{-1}0 \neq \emptyset$, there exists $u \in T^{-1}0$ such that $J_s u = u$ for all $s > 0$. Then we have

$$\begin{aligned}
\|x_1 - u\| &= \|\alpha_0 x + (1-\alpha_0)\, y_0 - u\| \\
&\leqslant \alpha_0 \|x - u\| + (1-\alpha_0) \|y_0 - u\| \\
&\leqslant \alpha_0 \|x - u\| + (1-\alpha_0)(\delta_0 + \|J_{r_0} x_0 - u\|) \\
&\leqslant \alpha_0 \|x - u\| + (1-\alpha_0)(\delta_0 + \|x_0 - u\|) \\
&\leqslant \|x - u\| + \delta_0.
\end{aligned}$$

If $\|x_k - u\| \leqslant \|x - u\| + \sum_{i=0}^{k-1} \delta_i$ holds for some $k \in \mathbb{N} \setminus \{0\}$, we can similarly show $\|x_{k+1} - u\| \leqslant \|x - u\| + \sum_{i=0}^{k} \delta_i$. Therefore, from $\sum_{n=0}^{\infty} \delta_n < \infty$, $\{x_n\}$ is bounded. Hence $\{J_{r_n} x_n\}$ and $\{y_n\}$ are also bounded. Then, from $r_n \to \infty$, we obtain

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \left\| \frac{x_n - J_{r_n} x_n}{r_n} \right\| = 0.$$

Next we will show

$$\limsup_{n\to\infty} \langle x - Px,\, y_n - Px \rangle \leqslant 0, \tag{3.3}$$

where P is the metric projection of H onto $T^{-1}0$. To prove this, it is sufficient to show

$$\limsup_{n\to\infty} \langle x - Px,\, J_{r_n} x_n - Px \rangle \leqslant 0$$

because $y_n - J_{r_n} x_n \to 0$. Now there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\lim_{i\to\infty} \langle x - Px,\, J_{r_{n_i}} x_{n_i} - Px \rangle = \limsup_{n\to\infty} \langle x - Px,\, J_{r_n} x_n - Px \rangle.$$

Since $\{J_{r_n} x_n\}$ is bounded, we may assume that $\{J_{r_{n_i}} x_{n_i}\}$ converges weakly to some $v \in H$. Then it follows that $v \in T^{-1}0$. Indeed, since $A_{r_n} x_n \in T J_{r_n} x_n$ and T is monotone,

$$\langle z - J_{r_{n_i}} x_{n_i}, z' - A_{r_{n_i}} x_{n_i} \rangle \geqslant 0$$

whenever $z' \in Tz$. From $A_{r_n} x_n \to 0$, we obtain $\langle z - v, z' \rangle \geqslant 0$ whenever $z' \in Tz$. Hence, from the maximality of T, we have $v \in T^{-1}0$. Since P is the metric projection of H onto $T^{-1}0$, we obtain

$$\begin{aligned}
\limsup_{n \to \infty} \langle x - Px, J_{r_n} x_n - Px \rangle &= \lim_{i \to \infty} \langle x - Px, J_{r_{n_i}} x_{n_i} - Px \rangle \\
&= \langle x - Px, v - Px \rangle \\
&\leqslant 0.
\end{aligned}$$

Thus we get (3.3).

Let $\varepsilon > 0$. Then, by $\sum_{n=0}^{\infty} \delta_n < \infty$, (3.3) and $\alpha_n \to 0$, there exists $m \in \mathbb{N}$ such that

$$M \sum_{j=m}^{\infty} \delta_j \leqslant \frac{\varepsilon}{2}, \qquad \langle x - Px, y_n - Px \rangle \leqslant \frac{\varepsilon}{6} \qquad \text{and} \qquad \alpha_n \|x - Px\|^2 \leqslant \frac{\varepsilon}{6}$$

for all $n \geqslant m$, where $M = \sup_{n \in \mathbb{N}} (\delta_n + 2 \|x_n - Px\|)$. This implies

$$\begin{aligned}
\|x_{n+m+1} - Px\|^2 &= \|\alpha_{n+m} x + (1 - \alpha_{n+m})\, y_{n+m} - Px\|^2 \\
&= \alpha_{n+m}^2 \|x - Px\|^2 + (1 - \alpha_{n+m})^2 \|y_{n+m} - Px\|^2 \\
&\quad + 2\alpha_{n+m}(1 - \alpha_{n+m}) \langle x - Px, y_{n+m} - Px \rangle \\
&\leqslant \alpha_{n+m} \frac{\varepsilon}{2} + (1 - \alpha_{n+m})(\delta_{n+m} + \|J_{r_{n+m}} x_{n+m} - Px\|)^2 \\
&\leqslant \alpha_{n+m} \frac{\varepsilon}{2} + (1 - \alpha_{n+m})(\delta_{n+m} + \|x_{n+m} - Px\|)^2 \\
&\leqslant \alpha_{n+m} \frac{\varepsilon}{2} + (1 - \alpha_{n+m})(\delta_{n+m} M + \|x_{n+m} - Px\|^2) \\
&\leqslant \alpha_{n+m} \frac{\varepsilon}{2} + \delta_{n+m} M + (1 - \alpha_{n+m}) \|x_{n+m} - Px\|^2
\end{aligned}$$

for all $n \in \mathbb{N}$. By induction, we obtain

$$\|x_{n+m+1} - Px\|^2 \leqslant \left\{1 - \prod_{j=m}^{n+m} (1-\alpha_j)\right\} \frac{\varepsilon}{2} + M \sum_{j=m}^{n+m} \delta_j + \left\{\prod_{j=m}^{n+m} (1-\alpha_j)\right\} \|x_m - Px\|^2$$

for all $n \in \mathbb{N}$. This implies

$$\begin{aligned}\|x_{n+m+1} - Px\|^2 &\leqslant \frac{\varepsilon}{2} + M \sum_{j=m}^{n+m} \delta_j + \left\{\prod_{j=m}^{n+m} (1-\alpha_j)\right\} \|x_m - Px\|^2 \\ &\leqslant \frac{\varepsilon}{2} + M \sum_{j=m}^{n+m} \delta_j + \exp\left(-\sum_{j=m}^{n+m} \alpha_j\right) \|x_m - Px\|^2.\end{aligned}$$

Therefore, from $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have

$$\limsup_{n\to\infty} \|x_n - Px\|^2 = \limsup_{n\to\infty} \|x_{n+m+1} - Px\|^2 \leqslant \frac{\varepsilon}{2} + M \sum_{j=m}^{\infty} \delta_j \leqslant \varepsilon.$$

Since ε is arbitrary, we can conclude that $\{x_n\}$ converges strongly to Px. ∎

4. WEAK CONVERGENCE THEOREM

In this section, we discuss the weak convergence of the proximal point algorithm. The sequence $\{x_n\}$ is generated by

$$\begin{cases} x_0 = x \in H, \\ y_n \approx J_{r_n} x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\, y_n, \qquad n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$.

LEMMA 2. *Let $T: H \to 2^H$ be a maximal monotone operator and let P be the metric projection of H onto $T^{-1}0$. Let $x \in H$ and let $\{x_n\}$ be a sequence generated by* (4.1) *under criterion* (3.2), *where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. If $T^{-1}0 \neq \emptyset$, then $\{Px_n\}$ converges strongly to $v \in T^{-1}0$, which is a unique element of $T^{-1}0$ such that*

$$\lim_{n\to\infty} \|x_n - v\| = \inf\{\lim_{n\to\infty} \|x_n - u\| : u \in T^{-1}0\}.$$

Proof. Let u be an element of $T^{-1}0$. Then we have

$$\begin{aligned}\|x_{n+1}-u\| &= \|\alpha_n x_n+(1-\alpha_n)\, y_n-u\| \\ &\leqslant \alpha_n\, \|x_n-u\|+(1-\alpha_n)\, \|y_n-u\| \\ &\leqslant \alpha_n\, \|x_n-u\|+(1-\alpha_n)(\delta_n+\|J_{r_n}x_n-u\|) \\ &\leqslant \alpha_n\, \|x_n-u\|+(1-\alpha_n)(\delta_n+\|x_n-u\|) \\ &\leqslant \|x_n-u\|+\delta_n\end{aligned}$$

for all $n\in\mathbb{N}$. Hence, from $\sum_{n=0}^{\infty}\delta_n<\infty$ and Tan and Xu [17, Lemma 1], $g(u)=\lim_{n\to\infty}\|x_n-u\|$ exists. Then g is a continuous convex function and $g(u)\to\infty$ as $\|u\|\to\infty$. Hence g attains its infimum over $T^{-1}0$. Let $l=\inf\{g(u) : u\in T^{-1}0\}$ and $K=\{w\in T^{-1}0 : g(w)=l\}$. Fix $v\in K$. Since P is the metric projection of H onto $T^{-1}0$, we have $\|x_n-Px_n\|\leqslant\|x_n-v\|$ for all $n\in\mathbb{N}$ and hence $\limsup_{n\to\infty}\|x_n-Px_n\|\leqslant l$. Suppose that $\limsup_{n\to\infty}\|x_n-Px_n\|<l$. Then we can choose $a>0$ and $m\in\mathbb{N}$ such that $\|x_n-Px_n\|\leqslant l-a$ for all $n\geqslant m$. Therefore we have

$$\begin{aligned}\|x_{n+h+1}-Px_n\| &\leqslant \|x_n-Px_n\|+\sum_{i=n}^{n+h}\delta_i \\ &\leqslant l-a+\sum_{i=n}^{n+h}\delta_i\end{aligned}$$

for all $n\geqslant m$ and $h\in\mathbb{N}$. Hence we obtain

$$\begin{aligned}l &\leqslant \lim_{h\to\infty}\|x_h-Px_n\| \\ &= \lim_{h\to\infty}\|x_{n+h+1}-Px_n\| \\ &\leqslant l-a+\sum_{i=n}^{\infty}\delta_i\end{aligned}$$

for all $n\geqslant m$. Since $\sum_{n=0}^{\infty}\delta_n<\infty$, we have $l\leqslant l-a<l$. This is a contradiction. So we can conclude that $\limsup_{n\to\infty}\|x_n-Px_n\|=l$.

Next we will show $\lim_{n\to\infty}Px_n=v$. If not, there exists $\varepsilon>0$ such that for any $h\in\mathbb{N}$, $\|Px_{h'}-v\|\geqslant\varepsilon$ for some $h'\geqslant h$. Let $b>0$ such that

$$b<\sqrt{l^2+\frac{\varepsilon^2}{8}}-l.$$

Then we can take $h' \in \mathbb{N}$ such that

$$M \sum_{i=h'}^{\infty} \delta_i \leqslant \frac{\varepsilon^2}{8}, \qquad \|x_{h'} - Px_{h'}\| \leqslant l+b \qquad \text{and} \qquad \|x_{h'} - v\| \leqslant l+b,$$

where $M = \sum_{n=0}^{\infty} \delta_n + 2 \sup_{n \in \mathbb{N}} \|x_n - (Px_n + v)/2\|$. Therefore we have

$$\begin{aligned}
\left\| x_{n+h'+1} - \frac{Px_{h'} + v}{2} \right\|^2 &\leqslant \left(\left\| x_{h'} - \frac{Px_{h'} + v}{2} \right\| + \sum_{i=h'}^{n+h'} \delta_i \right)^2 \\
&\leqslant \left\| x_{h'} - \frac{Px_{h'} + v}{2} \right\|^2 + M \sum_{i=h'}^{n+h'} \delta_i \\
&= 2 \left\| \frac{x_{h'} - Px_{h'}}{2} \right\|^2 + 2 \left\| \frac{x_{h'} - v}{2} \right\|^2 \\
&\quad - \left\| \frac{Px_{h'} - v}{2} \right\|^2 + M \sum_{i=h'}^{n+h'} \delta_i \\
&\leqslant 2 \left(\frac{l+b}{2} \right)^2 + 2 \left(\frac{l+b}{2} \right)^2 - \frac{\varepsilon^2}{4} + M \sum_{i=h'}^{n+h'} \delta_i \\
&= (l+b)^2 - \frac{\varepsilon^2}{4} + M \sum_{i=h'}^{n+h'} \delta_i
\end{aligned}$$

for all $n \in \mathbb{N}$. This implies

$$\begin{aligned}
l^2 &\leqslant \lim_{n \to \infty} \left\| x_{n+h'+1} - \frac{Px_{h'} + v}{2} \right\|^2 \\
&\leqslant (l+b)^2 - \frac{\varepsilon^2}{4} + M \sum_{i=h'}^{\infty} \delta_i \\
&\leqslant (l+b)^2 - \frac{\varepsilon^2}{8} \\
&< l^2.
\end{aligned}$$

This is a contradiction. Therefore $\{Px_n\}$ converges strongly to $v \in T^{-1}0$. Consequently v is a unique element of $T^{-1}0$ such that $g(v) = \inf\{g(u) : u \in T^{-1}0\}$. ∎

THEOREM 3. *Let $T: H \to 2^H$ be a maximal monotone operator and let P be the metric projection of H onto $T^{-1}0$. Let $x \in H$ and let $\{x_n\}$ be a sequence generated by* (4.1) *under criterion* (3.2), *where $\{\alpha_n\} \subset [0, 1]$ and*

$\{r_n\} \subset (0, \infty)$ satisfy $\alpha_n \in [0, k]$ for some k with $0 < k < 1$ and $\lim_{n \to \infty} r_n = \infty$. If $T^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to $v \in T^{-1}0$, where $v = \lim_{n \to \infty} Px_n$.

Proof. As in the proof of Lemma 2, $\lim_{n \to \infty} \|x_n - u\|$ exists for all $u \in T^{-1}0$ and, in particular, $\{x_n\}$ is bounded. Therefore there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some $v \in H$. We will prove $v \in T^{-1}0$. We first show $\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0$. In fact, we have

$$
\begin{aligned}
&(1-k)\, \alpha_n^2 \, \|x_n - y_n\|^2 \\
&\quad \leqslant (1-\alpha_n)\, \alpha_n \, \|x_n - y_n\|^2 \\
&\quad = \alpha_n \, \|x_n - u\|^2 + (1-\alpha_n)\, \|y_n - u\|^2 - \|x_{n+1} - u\|^2 \\
&\quad \leqslant \alpha_n \, \|x_n - u\|^2 + (1-\alpha_n)(\delta_n + \|J_{r_n} x_n - u\|)^2 - \|x_{n+1} - u\|^2 \\
&\quad \leqslant \alpha_n \, \|x_n - u\|^2 + (1-\alpha_n)(\delta_n + \|x_n - u\|)^2 - \|x_{n+1} - u\|^2 \\
&\quad \leqslant \alpha_n \, \|x_n - u\|^2 + (1-\alpha_n)(\delta_n M + \|x_n - u\|^2) - \|x_{n+1} - u\|^2 \\
&\quad \leqslant \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + \delta_n M,
\end{aligned}
$$

where $M = \sup_{n \in \mathbb{N}} (\delta_n + 2\, \|x_n - u\|)$. Therefore $\lim_{n \to \infty} \|x_{n+1} - y_n\| = \lim_{n \to \infty} \alpha_n \|x_n - y_n\| = 0$. Then we may also assume that $y_{n_i} \rightharpoonup v$ and hence $J_{r_{n_i}} x_{n_i} \rightharpoonup v$ because $y_n - J_{r_n} x_n \to 0$. Since $A_{r_n} x_n \in T J_{r_n} x_n$ and T is monotone,

$$
\langle z - J_{r_{n_i}} x_{n_i}, z' - A_{r_{n_i}} x_{n_i} \rangle \geqslant 0 \tag{4.2}
$$

holds whenever $z' \in Tz$. Since $r_n \to \infty$, we have

$$
\lim_{n \to \infty} \|A_{r_n} x_n\| = \lim_{n \to \infty} \left\| \frac{x_n - J_{r_n} x_n}{r_n} \right\| = 0.
$$

Tending i to ∞ in (4.2), we obtain

$$
\langle z - v, z' \rangle \geqslant 0
$$

for all z, z' with $z' \in Tz$. Then the maximality of T implies $v \in T^{-1}0$.

From Lemma 2, $\{Px_n\}$ converges strongly to some $v' \in T^{-1}0$. Since P is the metric projection of H onto $T^{-1}0$, we have

$$
\langle x_{n_i} - Px_{n_i}, w - Px_{n_i} \rangle \leqslant 0
$$

for all $w \in T^{-1}0$. Then we have

$$
\langle v - v', w - v' \rangle \leqslant 0
$$

for all $w \in T^{-1}0$. Putting $w = v$, we obtain $\|v - v'\|^2 \leqslant 0$ and hence $v = v'$. This implies that each weak subsequential limit of $\{x_n\}$ is equal to the strong limit of $\{Px_n\}$. Therefore $\{x_n\}$ converges weakly to $v \in T^{-1}0$, where $v = \lim_{n \to \infty} Px_n$. ∎

Next we study the rate of convergence of (4.1). According to Rockafellar [14], T^{-1} is said to be Lipschitz continuous at $0 \in H$ with modulus $a \geqslant 0$ if there exists a unique solution z_0 to $0 \in Tz$ (i.e., $T^{-1}0 = \{z_0\}$) and for some $\tau > 0$, we have

$$\|z - z_0\| \leqslant a \, \|w\| \tag{4.3}$$

whenever $z \in T^{-1}w$ and $\|w\| \leqslant \tau$. Rockafellar [14, Theorem 2] showed that if T^{-1} is Lipschitz continuous at 0 and $r_n \to \infty$, then the rate of convergence of (1.1) is superlinear. Then, using Rockafellar's method, we obtain the following result.

THEOREM 4. *Let $T: H \to 2^H$ be a maximal monotone operator. Let $\{x_n\}$ be a sequence generated by* (4.1) *under criterion*

$$\|y_n - J_{r_n} x_n\| \leqslant \gamma_n \, \|y_n - x_n\|,$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{\gamma_n\} \subset [0, \infty)$ satisfy $\alpha_n \in [0, k]$ for some k with $0 < k < 1$, $\lim_{n \to \infty} r_n = \infty$ and $\lim_{n \to \infty} \gamma_n = 0$. If $\{x_n\}$ is bounded and T^{-1} is Lipschitz continuous at 0 with modulus $a \geqslant 0$, then $\{x_n\}$ converges strongly to $v = T^{-1}0$. Moreover there exists an integer $N > 0$ such that

$$\|x_{n+1} - v\| \leqslant \theta_n \, \|x_n - v\|$$

for all $n \geqslant N$, where

$$\mu_n = \frac{a}{\sqrt{a^2 + r_n^2}}, \qquad \theta_n = \alpha_n + \frac{(1 - \alpha_n)(\mu_n + \gamma_n)}{1 - \gamma_n} \qquad \textit{and} \qquad 0 \leqslant \theta_n < 1$$

for all $n \geqslant N$.

Proof. Since T^{-1} is Lipschitz continuous at 0 with modulus $a \geqslant 0$, for some $\tau > 0$, we have $\|z - v\| \leqslant a \, \|w\|$ whenever $z \in T^{-1}w$ and $\|w\| \leqslant \tau$. From $\|J_{r_n} x_n - v\| \leqslant \|x_n - v\|$, $\{J_{r_n} x_n\}$ is bounded. Hence we obtain $A_{r_n} x_n \to 0$. Then there exists an integer $N > 0$ such that $\|A_{r_n} x_n\| \leqslant \tau$ and $\theta_n = \alpha_n + (1 - \alpha_n)(\mu_n + \gamma_n)/(1 - \gamma_n) < 1$ for all $n \geqslant N$. Since $J_{r_n} x_n \in T^{-1} A_{r_n} x_n$, we have

$$\|J_{r_n} x_n - v\| \leqslant a \, \|A_{r_n} x_n\| \tag{4.4}$$

for all $n \geqslant N$. It follows from (2.1) that

$$\|J_{r_n} x_n - v\|^2 + r_n^2 \|A_{r_n} x_n\|^2 \leqslant \|x_n - v\|^2. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$\|J_{r_n} x_n - v\| \leqslant \frac{a}{\sqrt{a^2 + r_n^2}} \|x_n - v\|$$

for all $n \geqslant N$. Therefore we have

$$\begin{aligned}
\|y_n - v\| &\leqslant \|y_n - J_{r_n} x_n\| + \|J_{r_n} x_n - v\| \\
&\leqslant \gamma_n \|y_n - x_n\| + \frac{a}{\sqrt{a^2 + r_n^2}} \|x_n - v\| \\
&\leqslant \gamma_n \|y_n - v\| + \gamma_n \|x_n - v\| + \mu_n \|x_n - v\|
\end{aligned}$$

for all $n \geqslant N$. This implies

$$\|y_n - v\| \leqslant \frac{\mu_n + \gamma_n}{1 - \gamma_n} \|x_n - v\|$$

for all $n \geqslant N$. Hence we obtain

$$\begin{aligned}
\|x_{n+1} - v\| &\leqslant \alpha_n \|x_n - v\| + (1 - \alpha_n) \|y_n - v\| \\
&\leqslant \left(\alpha_n + \frac{(1 - \alpha_n)(\mu_n + \gamma_n)}{1 - \gamma_n}\right) \|x_n - v\| \\
&= \theta_n \|x_n - v\|
\end{aligned}$$

for all $n \geqslant N$. This completes the proof. ∎

This theorem shows that the rate of convergence of (4.1) is linear and if $\lim_{n \to \infty} \alpha_n = 0$ then the rate is superlinear.

5. APPLICATIONS TO MINIMIZATION PROBLEM

In this section, we investigate our algorithms in the case of $T = \partial f$, where f is a proper lower semicontinuous convex function. Our discussion follows

Rockafellar [14, Section 4]. If $T=\partial f$, the algorithm (3.1) is reduced to the following:

$$\begin{cases} x_0 = x \in H, \\ y_n \approx \operatorname*{argmin}_{z \in H} \left\{ f(z) + \frac{1}{2r_n} \|z - x_n\|^2 \right\}, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)\, y_n, \qquad n \in \mathbb{N}, \end{cases} \tag{5.1}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. Here we consider the following criterion:

$$d(0, S_n(y_n)) \leqslant \frac{\delta_n}{r_n}, \tag{5.2}$$

where $\sum_{n=0}^{\infty} \delta_n < \infty$, $S_n(z) = \partial f(z) + (z - x_n)/r_n$ and $d(0, A) = \inf\{\|x\| : x \in A\}$. About (5.2), the following lemma was proved in Rockafellar [14, Proposition 3]

LEMMA 5. *If y_n is chosen according to criterion* (5.2), *then*

$$\|y_n - J_{r_n} x_n\| \leqslant \delta_n$$

holds, where $J_{r_n} = (I + r_n\, \partial f)^{-1}$.

THEOREM 6. *Let $f: H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Let $x \in H$ and let $\{x_n\}$ be a sequence generated by* (5.1) *under criterion* (5.2), *where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} r_n = \infty$. *If $(\partial f)^{-1} 0 \neq \emptyset$, then $\{x_n\}$ converges strongly to $v \in H$, which is the minimizer of f nearest to x. Further we have*

$$f(x_{n+1}) - f(v) \leqslant \alpha_n (f(x) - f(v)) + \frac{1-\alpha_n}{r_n} \|y_n - v\| \,(\delta_n + \|y_n - x_n\|).$$

Proof. Putting $g_n(z) = f(z) + \|z - x_n\|^2 / 2r_n$, we obtain

$$\partial g_n(z) = \partial f(z) + \frac{1}{r_n}(z - x_n) = S_n(z)$$

for all $z \in H$ and

$$J_{r_n} x_n = (I + r_n\, \partial f)^{-1} x_n = \operatorname*{argmin}_{z \in H} g_n(z).$$

It follows from Theorem 1 and Lemma 5 that $\{x_n\}$ converges strongly to $v \in H$ and $f(v) = \min_{z \in H} f(z)$. Since $\partial g_n(y_n)$ is a nonempty closed convex set, we can find the unique element w_n of $\partial g_n(y_n)$ nearest to the origin. Then we have

$$w_n - \frac{1}{r_n}(y_n - x_n) \in \partial f(y_n)$$

and

$$\|w_n\| \leqslant \frac{\delta_n}{r_n}. \tag{5.3}$$

The definition of subdifferential yields

$$f(v) \geqslant f(y_n) + \left\langle v - y_n, w_n - \frac{1}{r_n}(y_n - x_n) \right\rangle. \tag{5.4}$$

From (5.3) and (5.4), we obtain

$$\begin{aligned} f(x_{n+1}) - f(v) &= f(\alpha_n x + (1-\alpha_n) y_n) - f(v) \\ &\leqslant \alpha_n (f(x) - f(v)) + (1-\alpha_n)(f(y_n) - f(v)) \\ &\leqslant \alpha_n (f(x) - f(v)) + (1-\alpha_n) \left\langle y_n - v, w_n - \frac{1}{r_n}(y_n - x_n) \right\rangle \\ &\leqslant \alpha_n (f(x) - f(v)) + (1-\alpha_n) \|y_n - v\| \left(\|w_n\| + \frac{1}{r_n} \|y_n - x_n\| \right) \\ &\leqslant \alpha_n (f(x) - f(v)) + \frac{1-\alpha_n}{r_n} \|y_n - v\| \, (\delta_n + \|y_n - x_n\|). \end{aligned}$$

This completes the proof. ▮

Similarly we can show the following theorem concerning (4.1). Compare this result with Theorem 6.

THEOREM 7. *Let $f\colon H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Let $x \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x, \\ y_n \approx \operatorname*{argmin}_{z \in H} \left\{ f(z) + \dfrac{1}{2r_n} \|z - x_n\|^2 \right\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \qquad n \in \mathbb{N} \end{cases}$$

under criterion (5.2), *where* $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy* $\alpha_n \in [0, k]$ *for some* k *with* $0 < k < 1$ *and* $\lim_{n\to\infty} r_n = \infty$. *If* $(\partial f)^{-1} 0 \neq \emptyset$ *and* $\{u_n\}$ *is a sequence of points of* $(\partial f)^{-1} 0$ *nearest to* x_n, *then* $\{x_n\}$ *converges weakly to* $v \in H$, *which is the minimizer of* f *and satisfies* $v = \lim_{n\to\infty} u_n$. *Further we have*

$$f(x_{n+1}) - f(v) \leqslant \alpha_n(f(x_n) - f(v)) + \frac{1-\alpha_n}{r_n} \|y_n - v\| \, (\delta_n + \|y_n - x_n\|).$$

Proof. As in the proof of Theorem 6, we can prove this theorem. ∎

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